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Equivalence and Gauge in the Planck-Scale Aether Model

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Following the 19th century program by Dirichlet, Helmholtz, Thomson, and Hertz to obtain a completely kinematic interpretation of classical mechanics by the nonlinear Euler equations, an attempt is made to interpret the gauge and equivalence principles hydrodynamically in the framework of the Planck aether model.

1. INTRODUCTION

Following the successful reduction of the laws of thermodynamics to classical Newtonian mechanics, attempts were made by several leading 19th century scientists, notably Dirichlet, Helmholtz, Thomson, and Hertz, to reduce the laws of classical mechanics to the kinematics of a hypothetical aether described by the nonlinear Euler equation of an incompressible frictionless fluid.[2] These attempts preceded Einstein's program to give these laws a geometric meaning. Conceptually, kinematic and geometric laws are quite similar, suggesting that the pre-Einstein approach for a kinematic interpretation is ultimately not too far from Einstein's approach for a geometric interpretation. In the kinematic interpretation the dynamical laws of Newtonian mechanics are reduced to boundary conditions at the surface of the vortex cores formed in the frictionless incompressible aether, whereas in general relativity they are reduced to the geodesic motion in a curved space-time. Thomson showed that small-amplitude waves propagating through a vortex lattice formed in this aether can describe the electromagnetic waves of Maxwell's equations. With Maxwell's equations Lorentz invariance followed as

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a dynamic symmetry for objects held together by electromagnetic forces. But a problem common to all those aether models was that the aether must be massless, because an aether endowed with mass would lead to gravitational fields which are not observed. A solution for this problem was given by Einstein, who showed that Lorentz invariance could be understood as a symmetry of a four-dimensional space-time continuum, eliminating the need for an aether altogether.

The path taken by Einstein received considerable support through the success of his general theory of relativity (which generalized the Lorentz transformations between inertial frames of reference to those between noninertial frames) because it led to an unexpected solution of the long-standing problem of finding the correct equations for the gravitational field, where others before him, notably Maxwell, had failed. Notwithstanding Einstein's success, the idea of an aether sustained its appeal, as the only physically simple way to understand how a wave can propagate through a vacuum void of ordinary matter, which in the absence of a medium filling this vacuum is not possible in the absolute space of Newtonian mechanics.

The hypothetical aether of classical physics eliminated by Einstein in his special theory of relativity somehow reemerged in quantum mechanics as the zero-point energy of the vacuum. This zero-point energy has a divergent ω^3-frequency spectrum, the only one invariant under a Lorentz transformation. Therefore, unless this zero-point energy is cut off at some high frequency (resp. small length), it would lead to an aether of infinite mass density. Invoking general relativity by cutting it off at its Schwarzschild radius of $\sim 10^{-33}$ cm, which is equal the Planck length, the vacuum mass density would be of the order $\sim 10^{95}$ g/cm^3. Because this huge mass density would manifest itself in very large gravitational fields (obviously not observed), there must be something fundamentally wrong in our thinking.

Whereas in special relativity the need for an aether is eliminated, a kind of aether can be introduced in general relativity through the cosmological constant in Einstein's gravitational field equations, which acts like a pressure, and hence like a pressure-generating medium. Since the quantum mechanical zero-point energy of the vacuum has just the property of such a medium, it leads to a cosmological constant. Empirically, the cosmological constant is very small, if not exactly equal to zero, whereas the quantum mechanical zero-point energy predicts a very large cosmological constant.

According to Weinberg (1989, 1992) the smallness, if not vanishing, of the cosmological constant, remains one of the outstanding unsolved problems both of elementary particle physics and of cosmology. One way by which this problem could be solved is supersymmetry, where the divergent zero-point energy contributions from the boson and fermion fields can be made to cancel, but supersymmetry is not realized in nature, at least not at the low energies of everyday life. It does not really help if supersymmetry is broken below some high energy, because it would leave the zero-point energy uncompensated up to this energy. At best, it might be broken below ~ 100 GeV (the highest energy attainable with presently available particle accelerators), but even then it still would imply a vacuum mass density of $\sim 10^{14}$ g/cm^3 with a correspondingly large cosmological constant. This mass density is of the same order of magnitude as the mass density produced by the static nonvanishing Higgs field in the spontaneously broken gauge theory of the WSG model, which, too, should lead to large gravitational fields.

In an attempt which is reminiscent of past attempts to explain the various null results of prerelativity physics by a number of miraculous cancellations, each requiring a different mechanism, a miraculous cancellation of the different contributions to the vacuum energy coming from the zero-point and Higgs-field energies of various fields, some of them associated with hypothetical particles which have never been observed, has been proposed. In Einstein's special relativity all the miraculous cancellations were explained to result from one underlying universal principle. In light of this historical precedent, one should expect that the vanishing of the cosmological constant should rather be contained in the structure of the underlying fundamental field equation from which all elementary particles and their interactions are to be derived, and not be the result of a number of miraculous cancellations.[3]

Even though supersymmetric theories may not appear to be realized in nature, they can nevertheless provide us with a hint for the ultimately correct theory. In these theories the positive zero-point energy of a boson field is compensated by a negative energy contribution from a fermion field, with which it is supersymmetrically associated, and one might expect a likewise cancellation from two fields in the final theory. The reason for the cancellation in supersymmetric theories is that the Dirac equation possesses, besides its positive energy solution, also those of negative energy. It was shown by Schrödinger (1930, 1931) that because these negative energy states mix with those of positive energy, a Dirac particle executes a "*Zitterbewegung*" (quivering motion) explaining its spin as the angular momentum of this motion. The interpretation of this "*Zitterbewegung*" by Hönl and Papapetrou (1939*a,b*, 1940) and Bopp (1943, 1946, 1948) was that through the admixture of negative energy states, and hence negative mass states, a Dirac spinor can be described as a mass pole with a superimposed positive–negative mass dipole (pole–dipole particle). Such a pole–dipole particle moves on a helical trajectory where it reaches the velocity of light, and it is this helical motion which is Schrödinger's "*Zitterbewegung*." The negative energy states of the Dirac equation, though, cannot be directly observed, because all the negative

[3]This view has been expressed by Feynman (1988).

energy states are occupied with antiparticles interpreted as holes in a "sea" of occupied negative energy states. The "*Zitterbewegung*" analysis strongly supports the physical reality of negative masses, but the hypothesis for the existence of negative masses is not without problems. One reason is that an electrically charged particle having a negative mass could become self-accelerating by gaining negative kinetic energy through the emission of electromagnetic radiation. The same would happen even for an uncharged negative mass particle by the emission of gravitational radiation. A mass dipole formed from a positive and equal negative mass would be self-accelerating along a straight line, but this would not be the case for a pole–dipole particle, which follows a helical trajectory.

If the zero-point energy is cut off at its gravitational radius of $\sim 10^{-33}$ cm, it would lead to a vacuum densely occupied with Planck-mass black holes (Wheeler, 1968; Hawking, 1978). To compensate the huge mass density of $\sim 10^{95}$ g/cm^3 such an assembly would have, it was suggested by Sakharov (1968) that the vacuum might be filled with an equal number of compensating "ghost particles." However, for the compensation to work, these "ghost particles" must have a negative mass, resulting in unwanted self-accelerating runaway solutions. To overcome this difficulty, it has been proposed by the author (Winterberg, 1994) that the vacuum is densely filled with an equal number of positive and negative Planck masses but which (unlike in Sakharov's proposal) carry neither an electric nor a gravitational, or any other charge, and are for this reason not the source of long-range fields. It is rather assumed that they interact locally through contact-type delta-function potentials, very much as in Heisenberg's nonlinear spinor theory. Under this hypothesis all long-range fields are explained quantum mechanically through collective excitations of the hypothetical Planck aether, with the charge phenomenon having its cause in the zero-point fluctuations of the Planck masses. Quantizing the collective modes then leads to a spectrum of quasiparticles, representing the spectrum of observed elementary particles. If the collective modes obey the classical wave equation, special relativity follows as a dynamic symmetry for quasiparticles held together by forces transmitted through these waves. It is for this reason possible to assume that the Planck masses themselves are described by an exactly nonrelativistic law of motion. Because the particle number in a nonrelativistic theory is conserved, the Planck masses play the role of a kind of indestructible Leibnizian monads, with the property of the Leibnizian monads to possess "no windows," reflected in the property of the Planck masses not to be the source of any long-range field. Assuming that there is an equal number of positive and negative Planck masses, the average mass density of the Planck aether vanishes exactly, and with it the cosmological constant. The Planck aether can therefore accommodate the requirement of a massless aether, with the masslessness not in the absolute but only in the average. With the masslessness in the average only, wave propagation through the Planck aether is possible.

Mathematically, the model is described by two coupled nonlinear nonrelativistic operator field equations

$$i\hbar \frac{\partial \psi_{\pm}}{\partial t} = \mp \frac{\hbar^2}{2m_p} \nabla^2 \psi_{\pm} \pm 2\hbar c r_p^2 (\psi_{\pm}^{\dagger}\psi_{\pm} - \psi_{\mp}^{\dagger}\psi_{\mp})\psi_{\pm} \tag{1.1}$$

where m_p r_p are the Planck mass and Planck length, derived from the two Planck relations $Gm_p^2 = \hbar c$ and $m_p r_p c = \hbar$, where G is Newton's gravitational constant. The positive and negative mass components of the Planck aether, represented by the operators $\psi_{\pm}$, obey the canonical commutation relations

$$[\psi_{\pm}(\mathbf{r})\psi_{\pm}^{\dagger}(\mathbf{r}')] = \delta(\mathbf{r} - \mathbf{r}')$$

$$[\psi_{\pm}(\mathbf{r})\psi_{\pm}(\mathbf{r}')] = [\psi_{\pm}^{\dagger}(\mathbf{r})\psi_{\pm}^{\dagger}(\mathbf{r}')] = 0 \tag{1.2}$$

In the form given by (1.1) the proposed fundamental law resembles Heisenberg's nonlinear spinor field equation, except that (1.1) is exactly nonrelativistic. Unlike Heisenberg's theory, which had to assume an indefinite metric in Hilbert space, the state space constructed from (1.1) is always positive definite.[4]

Solutions for the quantized field equation (1.1) are in general difficult to obtain. But for a densely packed assembly of positive and negative Planck masses, with each mass component in a superfluid state represented by a completely symmetric wave function for equal Planck masses, a quite accurate picture of the possible solutions can be obtained from the Hartree–Fock approximation. There the field operators are replaced by their expectation values, with the product of three different field operators expressed by the product of the expectation values as follows ($\varphi = \langle\psi\rangle$, $\varphi^* = \langle\psi^{\dagger}\rangle$):

$$\langle\psi_{\pm}^{\dagger}\psi_{\pm}\psi_{\pm}\rangle \simeq 2\varphi_{\pm}^{*}\varphi_{\pm}^{2}$$

$$\langle\psi_{\mp}^{\dagger}\psi_{\mp}\psi_{\pm}\rangle \simeq \varphi_{\mp}^{*}\varphi_{\mp}\varphi_{\pm} \tag{1.3}$$

The factor 2 in the expectation value for the products of identical particles comes from the exchange interaction, which for the delta-function interaction potential is equal to the direct interaction. In the Hartree–Fock approximation, (1.1) therefore becomes

$$i\hbar \frac{\partial \varphi_{\pm}}{\partial t} = \mp \frac{\hbar^2}{2\mathrm{m}_p} \nabla^2 \varphi_{\pm} \pm 2\hbar c r_p^2 (2\varphi_{\pm}^{*}\varphi_{\pm} - \varphi_{\mp}^{*}\varphi_{\mp})\varphi_{\pm} \tag{1.4}$$

[4]The additional problem encountered by Heisenberg (1966) in his nonlinear spinor equation, which contains no mass term, was that there can be no interaction without a mass term. This problem does not arise for (1.1), which contains the Planck mass in a very fundamental way.

Making the substitutions

$$n_\pm = \varphi_\pm^* \varphi_\pm$$
$$n_\pm \mathbf{v}_\pm = \mp \frac{i\hbar}{2m_\mathrm{p}} [\varphi_\pm^* \nabla \varphi_\pm - \varphi_\pm \nabla \varphi_\pm^*] \tag{1.5}$$

one can bring (1.4) into its hydrodynamic form

$$\frac{\partial \mathbf{v}_\pm}{\partial t} + (\mathbf{v}_\pm \cdot \nabla)\mathbf{v}_\pm = -2c^2 r_\mathrm{p}^3 \nabla(2n_\pm - n_\mp) + \frac{1}{m_\mathrm{p}} \nabla Q_\pm$$
$$\frac{\partial n_\pm}{\partial t} + \nabla \cdot (n_\pm \mathbf{v}_\pm) = 0 \tag{1.6}$$

where

$$Q_\pm = \frac{\hbar^2}{2m_\mathrm{p}} \frac{\nabla^2 \sqrt{n_\pm}}{\sqrt{n_\pm}} \tag{1.7}$$

is the quantum potential, and where $n_\pm$ are the number of Planck masses per unit volume. In the vacuum ground state, it is assumed that $n_\pm^0 = 1/2r_\mathrm{p}^3$, consistent with the assumption of a vacuum densely packed with an equal number of positive and negative Planck masses.

In its linearized approximation, (1.6) leads to scalar compression waves propagating with the velocity of light. In addition, (1.6) has solutions which are quantized vortices, with the vortex core radius equal to a Planck length. Because the vacuum has an equal number of positive and negative Planck masses, a sponge of densely spaced positive and negative mass vortices can be formed from the ground state without the expenditure of energy. By drawing an analogy to classical hydrodynamics, a spacing of these vortex filaments about 10^3 times larger than their core radius is suggested. For the Planck aether, this leads to an energy scale about 10^3 times smaller than the Planck energy, in good agreement with the conjectured GUT scale of elementary particle physics.

A vortex sponge leads to two additional types of wave modes propagated by the vortex sponge lattice: an antisymmetric mode, which can describe Maxwell's electromagnetic waves, and a symmetric one, which can describe Einstein's gravitational waves. Because the filaments of the vortex lattice are coupled by the scalar compression waves, these transverse waves propagate with the velocity of light. Furthermore, a lattice of vortex rings has a resonant energy at $\sim \pm 10^{12}$ GeV, and resonances from the positive and negative masses of the vortex lattice can form excitonic solutions which have the property of Dirac spinors. Through the quantized vortices, charge can be explained to result from the zero-point fluctuations of Planck masses bound in the vortex filaments.

Even though both Maxwell's and Einstein's field equation can be derived from the Planck aether model, one would like to understand how the principle of equivalence (fundamental for Einstein's field equations) and the gauge principle (fundamental for gauge theories) can be understood within the framework of this model.

2. THE ORIGIN OF GRAVITATIONAL MASS

A Planck mass bound in a vortex filament has the quantum mechanical zero-point energy

$$E \sim \hbar c / r_{\mathrm{p}} \tag{2.1}$$

leading to a kinetic energy density within the filament given by

$$\varepsilon \sim \hbar c / r_{\mathrm{p}}^4 \tag{2.2}$$

The zero-point fluctuations generate a field of virtual phonons having their source in the Planck mass. If the strength of this phonon field is g, its energy density at the distance $r = r_{\mathrm{p}}$ is

$$\varepsilon \sim g^2 \tag{2.3}$$

hence

$$g \sim (\hbar c)^{1/2} / r_{\mathrm{p}}^2 = \sqrt{G} m_{\mathrm{p}} / r_{\mathrm{p}}^2 \tag{2.4}$$

the latter because $G m_{\mathrm{p}}^2 = \hbar c$. According to this result, Newton's law of gravitational attraction, and with it the property of gravitational mass, has its origin in the zero-point fluctuations of the Planck masses bound in the vortex filaments. For the attraction to make itself felt, both the attracting and attracted mass must be composed of Planck masses bound in vortex filaments. The gravitational field generated by a mass M, different from m_{p}, is the sum of all masses m_{p} bound in vortex filaments.[5] Its gravitational charge is $\sqrt{G}$ M and the gravitational field generated by it is

$$g = \sqrt{G} M / r^2 \tag{2.5}$$

The force exerted by g on another mass m (of charge $\sqrt{G} m$), which like M

[5] Arbitrarily small fractions of m_{p} are possible for an assembly consisting of an equal number of positive and negative Planck masses, with the positive kinetic energy of the positive Planck masses different from the absolute value of the negative kinetic energy of the negative Planck masses.

is composed of masses m_p bound in vortex filaments, is

$$F = \sqrt{G}m \times g = GmM/r^2 \tag{2.6}$$

The gravitational interaction therefore is the result of an attractive scalar phonon field in which the phonons propagate with the velocity of light. But because the vortex sponge propagates tensorial waves at larger wavelengths, simulating those derived from Einstein's gravitational field equations, the gravitational field appears to be tensorial in the low-energy limit.

3. THE ORIGIN OF INERTIAL MASS AND THE PRINCIPLE OF EQUIVALENCE

The equivalence of the gravitational and inertial masses can most easily be demonstrated for the limiting case of an incompressible Planck aether. To prove the principle of equivalence in this limit, we use the equations for an incompressible frictionless fluid

$$\begin{aligned} \operatorname{div} \mathbf{v} &= 0 \\ \rho\frac{d\mathbf{v}}{dt} &= -\operatorname{grad} p \end{aligned} \tag{3.1}$$

Applied to the positive mass component of (1.6), we have $\rho = nm_p$, $n = 1/2r_p^3$, hence $p = nm_pc^2$. In the limit of an incompressible fluid, the pressure p plays the role of a Lagrange multiplier, with which the incompressibility condition $\operatorname{div} \mathbf{v} = 0$ in the Lagrange density function has to be multiplied (Sommerfeld, 1950). The force density resulting from a pressure gradient is for this reason a constraint force. The same is true for the inertial forces in general relativity, where they are constraint forces imposed by curvilinear coordinates in a noninertial reference system.

We now show that the inertial force density $\rho\, d\mathbf{v}/dt$ in Euler's equation can be interpreted as a constraint force resulting from the interaction with the Planck masses filling all of space. Apart from those regions occupied by the vortex filaments, the Planck aether is everywhere superfluid and must obey the equation

$$\operatorname{curl} \mathbf{v} = 0 \tag{3.2}$$

With the incompressibility condition $\operatorname{div} \mathbf{v} = 0$, the solutions of Euler's equation for the superfluid regions are solutions of Laplace's equation for the velocity potential ψ

$$\nabla^2\psi = 0 \tag{3.3}$$

where $\mathbf{v} = -\operatorname{grad} \psi$. A solution of (3.3) is solely determined by the boundary conditions on the surface of the vortex filaments, as can be also seen more directly in the following way: With the help of the identity

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\frac{v^2}{2}\right) - \mathbf{v} \times \text{curl } \mathbf{v} \tag{3.4}$$

one can obtain an equation for p by taking the divergence on both sides of Euler's equation

$$\nabla^2\left(\frac{p}{\rho} + \frac{v^2}{2}\right) = \text{div}(\mathbf{v} \times \text{curl } \mathbf{v}) \tag{3.5}$$

Solving for p, one has (up to a function of time depending on the initial conditions, but which is otherwise of no interest)

$$\frac{p}{\rho} = -\frac{v^2}{2} - \int \frac{\text{div}(\mathbf{v} \times \text{curl } \mathbf{v})}{4\pi|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' \tag{3.6}$$

Inserting this expression for p into Euler's equation, it becomes an integrodifferential equation

$$\frac{d\mathbf{v}}{dt} = \nabla\left(\frac{v^2}{2}\right) + \nabla \int \frac{\text{div}(\mathbf{v} \times \text{curl } \mathbf{v})}{4\pi|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' \tag{3.7}$$

If the superfluid Planck aether could be set into uniform rotational motion of angular velocity ω, one would have

$$v = r\omega \tag{3.8}$$

and hence

$$|\nabla(v^2/2)| = r\omega^2 \tag{3.9}$$

but because for a superfluid curl $\mathbf{v} = 0$, the velocity field (3.8) is excluded. If set into uniform rotation, a superfluid rather sets up a lattice of parallel vortex filaments, possessing the same average vorticity as a uniform rotation. If space would be permeated by such an array of vortices, it would show up in an anisotropy which is not observed. Other nonrotational motions leading to nonvanishing $\nabla(v^2/2)$ terms involve expansions or dilations, excluded for an incompressible fluid. Therefore, the only remaining term on the r.h.s. of (3.7) is the integral term. It is nonlocal and purely kinematic. Through it a "field" is transmitted to the position $\mathbf{r}$ over the distance $|\mathbf{r} - \mathbf{r}'|$. One can therefore write for the incompressible Planck aether

$$\frac{d\mathbf{v}}{dt} = \nabla \int \frac{\text{div}(\mathbf{v} \times \text{curl } \mathbf{v})}{4\pi|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' \tag{3.10}$$

As Einstein's equation of motion for a test body, which is the geodesic in a curved four-dimensional space, equation (3.10) does not contain the mass of the test body. It is for this reason purely kinematic, giving a kinematic interpretation of inertial mass.

The integral of the r.h.s. of (3.10) must be extended over all regions where curl $\mathbf{v} \neq 0$. These are the regions occupied by vortex filaments. In the Planck aether, the vortex core radius is the Planck length r_p, with the vortex having the azimuthal velocity

$$\begin{aligned} v_\varphi &= c(r_p/r), \qquad r > r_p \\ &= 0, \qquad\qquad r < r_p \end{aligned} \tag{3.11}$$

Using Stokes' theorem for a surface cutting through the vortex core and having a radius equal to the core radius r_p, one finds that at $r = r_p$

$$\text{curl}_z \mathbf{v} = 2c/r_p \tag{3.12}$$

A length element of the vortex tube equal to r_p makes the contribution

$$\int_{r_p} \frac{\text{div}(\mathbf{v} \times \text{curl } \mathbf{v})}{4\pi|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' \simeq \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} \oint_{r_p} (\mathbf{v} \times \text{curl } \mathbf{v}) \cdot d\mathbf{f} \tag{3.13}$$

where $\int df \simeq 2\pi r_p^2$ is the surface element for a length r_p of the vortex tube. Therefore, each such element r_p makes the contribution

$$\int_{r_p} \frac{\text{div}(\mathbf{v} \times \text{curl } \mathbf{v})}{4\pi|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' \simeq \frac{r_p c^2}{|\mathbf{r} - \mathbf{r}'|} \tag{3.14}$$

The contribution to $d\mathbf{v}/dt$ of an element located at $\mathbf{r}'$ along $\mathbf{r} - \mathbf{r}'$ then is

$$\frac{d\mathbf{v}}{dt} = -\frac{r_p c^2}{|\mathbf{r} - \mathbf{r}'|^2} \tag{3.15}$$

Because of $Gm_p^2 = \hbar c$ and $m_p r_p c = \hbar$, this can be written as follows:

$$\frac{d\mathbf{v}}{dt} = -\frac{Gm_p}{|\mathbf{r} - \mathbf{r}'|^2} \tag{3.16}$$

demonstrating the origin of the inertial mass in the gravitational mass of all the Planck masses in the universe bound in vortex filaments.

To obtain a value for the inertial mass of a body, its interaction with all the Planck masses in the universe must be taken into account. By order of magnitude, this sum can be estimated by placing all these Planck masses at an average distance R, with R given by

$$R = GM/c^2 \tag{3.17}$$

and where M is the mass of the universe, equal to the sum of all Planck masses. Relation (3.17) follows from general relativity, but it can also be justified if one assumes that the total energy of the universe is zero, with the positive rest mass energy Mc^2 equal to the negative gravitational potential energy $-GM^2/R$. Summing up all contributions to $d\mathbf{v}/dt$, one obtains from (3.16) and (3.17)

$$\left|\frac{d\mathbf{v}}{dt}\right| \sim \frac{GM}{R^2} = \frac{c^2}{R} \tag{3.18}$$

as in Mach's principle. Because the gravitational interaction is solely determined by the gravitational mass, the equivalence of the inertial and gravitational mass is obvious.

We take notice that according to Mach's principle, inertia results from a "gravitational field" set up by an accelerated motion of all the masses in the universe relative to an observer. It is for this reason that the hypothetical gravitational field of Mach's principle must act instantaneously with an infinite speed. By contrast, the inertia in the Planck aether is due to the presence of the Planck masses filling all of space relative to which an absolute accelerated motion generates inertia as a purely local effect, not as a global effect as in Mach's principle. Notwithstanding this difference in explaining the origin of inertia in the Planck aether model and in Mach's principle, the latter can be recovered from the Planck aether model provided not only all the masses in the universe are set into an accelerated motion relative to an observer, but with them the Planck aether as well, or what is the same, the physical vacuum of quantum field theory. Only then is complete kinematical equivalence established, and only then would Mach's principle not require action at a distance.

The conclusions made for an incompressible Planck aether can be easily generalized to a compressible Planck aether, at least in the limit in which the equations of motions are linearized.

To show this, we start from the Euler and continuity equations of a compressible fluid

$$\left.\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\frac{v^2}{2}\right) - \mathbf{v} \times \operatorname{curl} \mathbf{v} &= -\frac{1}{\rho}\nabla p \\ \frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{v} &= 0 \end{aligned}\right\} \tag{3.19}$$

Linearizing with regard to p and ρ, by putting

$$\rho \to \rho_0 + \rho, \qquad p \to p_0 + p \tag{3.20}$$

where $\rho/\rho_0 \ll 1$ and $p/p_0 \ll 1$, one has

$$\left.\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\frac{v^2}{2}\right) - \mathbf{v} \times \text{curl } \mathbf{v} &= -\frac{1}{\rho_0}\nabla p \\ \frac{\partial \rho}{\partial t} + \rho_0 \text{ div } \mathbf{v} &= 0 \end{aligned}\right\} \tag{3.21}$$

where it was assumed that $\nabla\rho_0 = 0$. From the second of these equations, one has

$$\text{div } \mathbf{v} = -\frac{1}{\rho_0}\frac{\partial \rho}{\partial t} = -\frac{1}{\rho_0}\frac{\partial \rho}{\partial p}\frac{\partial p}{\partial t} = -\frac{1}{\rho_0 c^2}\frac{\partial p}{\partial t} \tag{3.22}$$

where because of $p = nm_p c^2 = \rho c^2$, one has $\partial p/\partial \rho = c^2$. Taking the divergence on both sides of the first of (3.21) leads to

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right)\frac{p}{\rho_0} = \text{div}\left[\mathbf{v} \times \text{curl } \mathbf{v} - \nabla\left(\frac{v^2}{2}\right)\right] \tag{3.23}$$

As before, we omit the $\nabla(v^2/2)$ term and obtain from (3.23) the retarded potential solution

$$\frac{p}{\rho_0} = -\int \frac{[\text{div}(\mathbf{v} \times \text{curl } \mathbf{v})]_{\text{ret}}}{4\pi|\mathbf{r} - \mathbf{r}'|}\, d\mathbf{r}' \tag{3.24}$$

Reinserted into Euler's equation, we finally have

$$\frac{d\mathbf{v}}{dt} = \nabla \int \frac{[\text{div}(\mathbf{v} \times \text{curl } \mathbf{v})]_{\text{ret}}}{4\pi|\mathbf{r} - \mathbf{r}'|}\, d\mathbf{r}' \tag{3.25}$$

taking the place of (3.10). Therefore, all previous results remain unchanged as long as no localized rapid large-scale changes in the cosmic matter distribution occur.

4. INTERPRETATION OF GAUGE INVARIANCE

In Maxwell's equations the electric and magnetic fields can be expressed through a scalar potential Φ and a vector potential $\mathbf{A}$:

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \text{grad } \Phi \tag{4.1}$$

$$\mathbf{H} = \text{curl } \mathbf{A}$$

$\mathbf{E}$ and $\mathbf{H}$ remain unchanged under the gauge transformation of the potentials

$$\Phi' = \Phi - \frac{1}{c}\frac{\partial f}{\partial t}$$

$$\mathbf{A}' = \mathbf{A} + \operatorname{grad} f \tag{4.2}$$

where f is called the gauge function. Imposing on Φ and $\mathbf{A}$ the Lorentz gauge condition

$$\frac{1}{c}\frac{\partial \Phi}{\partial t} + \operatorname{div} \mathbf{A} = 0 \tag{4.3}$$

the gauge function must satisfy the wave equation

$$-\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} + \nabla^2 f = 0 \tag{4.4}$$

In an electromagnetic field the force on a charge e is

$$F = e\left[\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{H}\right]$$

$$= e\left[-\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \operatorname{grad}\Phi + \frac{1}{c}\mathbf{v} \times \operatorname{curl}\mathbf{A}\right] \tag{4.5}$$

By making a gauge transformation of the Hamilton operator in the Schrödinger wave equation, the wave function transforms as

$$\Psi' = \Psi \exp\left[\frac{ie}{\hbar c} f\right] \tag{4.6}$$

leaving invariant the probability density $\Psi^*\Psi$.

To give gauge invariance a hydrodynamic interpretation, we compare (4.5) with the force acting on a test body of mass m placed into the moving Planck aether. This force follows from Euler's equation and is

$$F = m\frac{d\mathbf{v}}{dt} = m\left[\frac{\partial \mathbf{v}}{\partial t} + \operatorname{grad}\left(\frac{v^2}{2}\right) - \mathbf{v} \times \operatorname{curl}\mathbf{v}\right] \tag{4.7}$$

Complete analogy between (4.5) and (4.7) is established if one sets

$$\Phi = -\frac{m}{2e}v^2$$

$$\mathbf{A} = -\frac{mc}{e}\mathbf{v} \tag{4.8}$$

According to (4.2) and (4.6), Φ and **A** shift the phase of a Schrödinger wave by

$$\delta\varphi = \frac{e}{\hbar}\int_{t_1}^{t_2} \Phi\, dt$$

$$\delta\varphi = -\frac{e}{\hbar c}\oint \mathbf{A}\cdot d\mathbf{s} \tag{4.9}$$

The corresponding expressions for a gravitational field can be directly obtained from the equivalence principle (Hund, 1948). If $\partial\mathbf{v}/\partial t$ is the acceleration and $\boldsymbol{\omega}$ the angular velocity of the universe relative to a reference system assumed to be at rest, the inertial forces in this system are

$$\mathbf{F} = m\left[\frac{\partial\mathbf{v}}{\partial t} + \dot{\boldsymbol{\omega}} \times \mathbf{r} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - \dot{\mathbf{r}} \times 2\boldsymbol{\omega}\right] \tag{4.10}$$

For (4.10) we write

$$\mathbf{F} = m\left[\hat{\mathbf{E}} + \frac{1}{c}\mathbf{v} \times \hat{\mathbf{H}}\right] \tag{4.11}$$

where

$$\hat{\mathbf{E}} = \frac{\partial\mathbf{v}}{\partial t} + \dot{\boldsymbol{\omega}} \times \mathbf{r} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

$$\hat{\mathbf{H}} = -2c\boldsymbol{\omega} \tag{4.12}$$

With

$$\operatorname{curl}(\dot{\boldsymbol{\omega}} \times \mathbf{r}) = 2\dot{\boldsymbol{\omega}}$$

$$\operatorname{div}(-\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})) = 2\omega^2 \tag{4.13}$$

one has

$$\operatorname{div} \hat{\mathbf{H}} = 0$$

$$\frac{1}{c}\frac{\partial\hat{\mathbf{H}}}{\partial t} + \operatorname{curl} \hat{\mathbf{E}} = 0 \tag{4.14}$$

$\hat{\mathbf{E}}$ and $\hat{\mathbf{H}}$ can be derived from a scalar and vector potential

$$\hat{\mathbf{E}} = -\frac{1}{c}\frac{\partial\hat{\mathbf{A}}}{\partial t} - \operatorname{grad} \hat{\Phi}$$

$$\hat{\mathbf{H}} = \operatorname{curl} \hat{\mathbf{A}} \tag{4.15}$$

Applied to a rotating reference system, one has

$$\hat{\boldsymbol{\Phi}} = -\frac{1}{2}(\boldsymbol{\omega} \times \mathbf{r})^2$$
$$\hat{\mathbf{A}} = -c(\boldsymbol{\omega} \times \mathbf{r}) \tag{4.16}$$

or

$$\hat{\boldsymbol{\Phi}} = -\frac{v^2}{2}$$
$$\hat{\mathbf{A}} = -c\mathbf{v} \tag{4.17}$$

Apart from the factor m/e, this is the same as (4.8).

For weak gravitational fields produced by slowly moving matter, Einstein's linearized gravitational field equations permit the gauge condition (replacing the Lorentz gauge)

$$\frac{4}{c}\frac{\partial \Phi}{\partial t} + \operatorname{div} \mathbf{A} = 0$$
$$\frac{\partial \hat{\mathbf{A}}}{\partial t} = 0 \tag{4.18}$$

with the gauge transformation for Φ and $\mathbf{A}$

$$\Phi' = \Phi$$
$$\hat{\mathbf{A}}' = \hat{\mathbf{A}} + \operatorname{grad} f \tag{4.19}$$

where f has to satisfy the potential equation

$$\nabla^2 f = 0 \tag{4.20}$$

For a stationary gravitational field the vector potential changes the phase of the Schrödinger wave function according to

$$\Psi' = \Psi \exp\left[\frac{im}{\hbar c} f\right] \tag{4.21}$$

leading to a phase shift on a closed path

$$\delta\varphi = -\frac{m}{\hbar c}\oint \hat{\mathbf{A}} \cdot d\mathbf{s}$$
$$= \frac{m}{\hbar}\oint \mathbf{v} \cdot d\mathbf{s} \tag{4.22}$$

5. COMPARISON OF THE PHASE SHIFTS CAUSED BY THE MAGNETIC AND GRAVITATIONAL VECTOR POTENTIALS

In the hydrodynamic interpretation suggested by the Planck aether hypothesis, the phase shifts caused by either the magnetic or the gravitational vector potential result from a circular flow of the Planck aether. The principle of equivalence can precisely relate this circular flow to the angular velocity of a rotating platform. According to (4.17), one has for the gravitational vector potential in a rotating frame of reference

$$\hat{A} = -\omega c r \tag{5.1}$$

with to the phase shift given by (4.22). One can apply (4.22) to the Sagnac effect for photons of frequency ν, by putting $mc^2 = h\nu = 2\pi\hbar\nu$, with the result that

$$\begin{aligned} \delta\varphi &= (2\pi\nu/c^2) \oint \mathbf{v} \cdot d\mathbf{s} \\ &= 2\omega(\pi r^2) \cdot (2\pi\nu/c^2) \end{aligned} \tag{5.2}$$

the same as predicted without quantum mechanics.

The formula (4.22) can also be applied to a neutron interferometer placed on a rotating platform. An experiment of this kind, using the rotating earth as in the Michelson–Gale version of the Sagnac experiment, was actually carried out (Staudenmann, 1980), confirming the theoretically predicted phase shift.

We now compute the phase shift (4.9) by a magnetic vector potential. To make a comparison with the gravitational vector potential in the Sagnac effect, we consider the magnetic field produced by an infinitely long cylindrical solenoid of radius R. Inside the solenoid the field is constant, vanishing outside. If the magnetic field inside the solenoid is H, the vector potential is

$$\begin{aligned} A_\varphi &= \frac{1}{2} Hr, \qquad r < R \\ &= \frac{1}{2}\frac{HR^2}{r}, \qquad r > R \end{aligned} \tag{5.3}$$

According to (4.9), the vector potential on a closed path leads to the phase shift

$$\begin{aligned} \delta\varphi &= -\frac{e}{\hbar c} H\pi r^2, \qquad r < R \\ &= -\frac{e}{\hbar c} H\pi R^2, \qquad r > R \end{aligned} \tag{5.4}$$

As noted by Aharonov and Bohm (1959), there is a phase shift for $r > R$, even though for $r > R$, $H = 0$ (because for $r > R$, curl $\mathbf{A} = 0$).

Expressing $\mathbf{A}$ by (4.8) through v, the hypothetical circular aether velocity,

$$v_\varphi = -\frac{e}{2mc} Hr, \qquad r < R$$

$$= -\frac{e}{2mc}\frac{HR^2}{r}, \qquad r > R \tag{5.5}$$

one sees that inside the coil the velocity profile is the same as in a rotating frame of reference, having outside the coil the form of a potential vortex. If expressed in terms of the aether velocity, the phase shift becomes

$$\delta\varphi = \frac{m}{\hbar}\oint \mathbf{v}\cdot d\mathbf{s} \tag{5.6}$$

the same as (4.22) for the vector potential created by a gravitational field, and hence the same as in the Sagnac experiment and neutron interference experiment. But for the magnetic vector potential the aether velocity can easily become much larger than in any rotating platform experiment. According to (5.5), the velocity reaches a maximum at $r = R$, where it is

$$\frac{|v_{\max}|}{c} = \frac{eHR}{2mc^2} \tag{5.7}$$

For electrons this is $|v_{\max}|/c \simeq 3 \times 10^{-4}\, HR$, where H is measured in Gauss. For $H = 10^4$ G, this would mean that $v_{\max} \simeq c$ for $R \gtrsim 0.3$ cm.[6] If this would be the same aether velocity felt on a rotating platform, it would lead to an enormous centrifugal and Coriolis field inside the coil, obviously not observed.

The Planck aether model can give a simple explanation for this paradox. The Planck aether consists of two superfluid components, one composed of positive Planck masses and the other one of negative Planck masses. The two components can freely flow through each other, making possible two configurations, one where both components are corotating and one where they are counterrotating. The corotating configuration is realized on a rotating platform, where it leads to the Sagnac and neutron interference effects. This suggests that in the presence of a magnetic vector potential the two superfluid components are counterrotating. Outside the coil, where curl $\mathbf{A} = 0$, the

[6]It thus seems to follow that the aether can reach superluminal velocities for rather modest magnetic fields. In this regard it must be emphasized that in the Planck aether model all relativistic effects are explained dynamically, with the aether itself obeying an exactly nonrelativistic law of motion. The aether can for this reason assume superluminal velocities.

magnetic energy density vanishes, implying that the magnitudes of both velocities are exactly the same. Inside the coil, where curl $\mathbf{A} \neq 0$, there must be a small imbalance in the velocity of the positive over the negative Planck masses to result in a positive energy density.

In Maxwell's theory, the electric charge satisfies a conservation law, as does the number of the Planck masses. The electric charge can for this reason only reside in one species of Planck masses, for negative charges in the positive Planck masses, for positive charges in the negative Planck masses, or vice versa. Since the origin of the electric charge would still be the zero-point fluctuations of the Planck masses bound in the quantized vortex filaments, it becomes plausible why the electromagnetic coupling constant $e^2/\hbar c \simeq 1/137$ is not too far away from the gravitational coupling constant of the Planck masses $Gm_p^2/\hbar c = 1$. By comparison, the gravitational coupling constant of a Dirac spinor of mass m, $Gm^2/\hbar c$, is typically 44 orders of magnitude smaller. In the Planck aether model a Dirac spinor is an exciton formed from the large resonances, one having positive and the other one having a negative mass. Because the gravitational field couples to both positive and negative masses, the sum of which can be very small, the gravitational coupling constant for a Dirac spinor can for this reason become much smaller than its electromagnetic coupling constant.

6. ANALOGIES BETWEEN EINSTEIN-GRAVITY AND NON-ABELIAN GAUGE FIELD THEORIES

In Einstein's gravitational field theory the force on a particle is expressed by the Christoffel symbols. They are obtained from first-order derivatives of the ten potentials of the gravitational field represented by the ten components of the metric tensor. From the Christoffel symbols the Riemann curvature tensor is structured by the following symbolic equation

$$\mathbf{R} = \mathrm{Curl}\, \boldsymbol{\Gamma} + \boldsymbol{\Gamma} \otimes \boldsymbol{\Gamma} \tag{6.1}$$

The expression for the field strength, and hence force, in Yang–Mills field theories is symbolically given by (g is a coupling constant with the dimension of electric charge)

$$\mathbf{W} = \mathrm{Curl}\, \mathbf{A} - g^{-1}\mathbf{A} \otimes \mathbf{A} \tag{6.2}$$

It, too, has the form of a curvature tensor, albeit not in space-time, but in internal charge space, in QCD for example, in color space. It was Riemann who wondered if in the small there might be a departure from the Euclidean metric. The Yang–Mills field theories have answered this question in a quite unexpected way, not as a non-Euclidean structure in space-time, but rather as one in charge space, making itself felt only in the small.

Comparing (6.1) with (6.2), one can from a gauge-field-theoretic point of view consider the Γ^i_{kl} as gauge fields. If the curvature tensor vanishes, they can be globally eliminated by a transformation to a pseudo-Euclidean Minkowski space-time metric. One may for this reason call the Γ^i_{kl} pure gauge fields which for a vanishing curvature tensor can always be transformed away by a gauge transformation. Likewise, if the curvature tensor in (6.2) vanishes, one may globally transform away the gauge potentials.

From the Newtonian point of view, contained in Einstein's field equations, the force is always related to the first derivative of a potential. Apart from the nonlinear term in (6.2), this is also true for a Yang–Mills field theory. But with the inclusion of the nonlinear terms, (6.2) has also the structure of a curvature tensor. In Einstein's theory the curvature tensor involves second-order derivatives of the potentials, whereas in a Yang–Mills field theory the curvature tensor in charge space involves only first-order derivatives of the potentials. This demonstrates a displacement of the hierarchy for the potentials with regard to the forces. A displacement of hierarchies also occurs in fluid dynamics by comparing Newton's point particle dynamics with Helmholtz's line vortex dynamics (Sommerfeld, 1950). Whereas in Newton's point particle dynamics the equation of motion is $m\ddot{\mathbf{r}} = \mathbf{F}$, the corresponding equation in Helmholtz's vortex dynamics is $\mu\dot{\mathbf{r}} = \mathbf{F}$. Therefore, whereas in Newtonian mechanics a body moves with constant velocity in the absence of a force, it remains at rest in vortex dynamics. Whereas what is at rest remains undetermined in Newtonian mechanics, it is fully determined in vortex dynamics, where at rest means at rest with regard to the fluid. The same would have to be true with regard to the hypothetical Planck aether.

The hydrodynamics of the Planck aether model suggests that the hierarchical displacement of the curvature tensor for Einstein and Yang–Mills fields is related to the hierarchical displacement of the vortex equation of motion if compared with the Newtonian equation of motion. This idea can be explored a little further. To do this we consider the force between two magnetic dipoles separated by the distance r. Their dipole moments $\mathbf{m}_1$ and $\mathbf{m}_2$ have the magnetic vector potentials

$$\mathbf{A}_1 = \frac{\mathbf{m}_1 \times \mathbf{e}_r}{r^2}, \qquad \mathbf{A}_2 = -\frac{\mathbf{m}_2 \times \mathbf{e}_r}{r^2} \tag{6.3}$$

where $\mathbf{e}_r$ is perpendicular to $\mathbf{m}_1$ and $\mathbf{m}_2$. The magnetic force on $\mathbf{m}_2$ by $\mathbf{m}_1$ is given by

$$F = \nabla(\mathbf{m}_2 \cdot \text{curl } \mathbf{A}_1) = 6\mathbf{A}_1 \cdot \mathbf{A}_2 \mathbf{e}_r \tag{6.4}$$

showing the origin of the nonlinear terms quadratic in the vector potentials, typical for Yang–Mills theories.

Table I. Hierarchical Displacement of Einstein and Yang–Mills Field Theories as Related to the Hierarchical Displacement of Newton Point-Particle and Helmholtz Line-Vortex Dynamics.

Newton point-particle dynamics and Einstein's gravitational field theory		Kinematic quantities	Helmholtz line-vortex dynamics and Yang–Mills field theories	
		$\mathbf{r}$	ψ	Velocity potential
			f	Gauge functions
Newtonian potential	ϕ	$\dot{\mathbf{r}}$	$-\nabla\psi$	Force on line vortex
Metric tensor	g_{ik}		A	Gauge potentials
Force on point particle	$-\nabla\phi$	$\ddot{\mathbf{r}}$	$\mathbf{W} = \text{Curl } \mathbf{A} - \mathbf{g}^{-1}\mathbf{A} \otimes \mathbf{A}$	Yang–Mills force field expressed by charge-space curvature tensor
Gravitational force field expressed by Christoffel symbols	$\mathbf{\Gamma}$			
Einstein's field equations expressed by metric-space curvature tensor	$\mathbf{R} = \text{Curl } \mathbf{\Gamma} + \mathbf{\Gamma} \otimes \mathbf{\Gamma}$			

Because of the analogy between magnetic fields generated by current filaments, and velocity fields generated by vortex filaments (first recognized by Helmholtz), one has for a current filament of current density $\mathbf{j}$

$$\mathbf{A} = \frac{1}{c}\int \frac{\mathbf{j}}{r}\, d\mathbf{r} \tag{6.5}$$

and for a vortex filament of vorticity $\boldsymbol{\omega}$

$$\hat{\mathbf{A}} = \frac{1}{2\pi}\int \frac{\boldsymbol{\omega}}{r}\, d\mathbf{r} \tag{6.6}$$

From (6.5) one obtains $\mathbf{H} = \text{curl } \mathbf{A}$, and from (6.6) $\mathbf{v} = \text{curl } \hat{\mathbf{A}}$. With the electric current density $\mathbf{j}$ and vorticity $\boldsymbol{\omega}$ related to each other by $\mathbf{j} = (c/2\pi)\boldsymbol{\omega}$, one has $j = (c/4\pi)$ curl curl $\mathbf{A}$, and $\boldsymbol{\omega} = (1/2)$ curl curl $\hat{\mathbf{A}}$.

The magnetic moment of a current loop of radius R and carrying the current I is

$$m = I\pi R^2/c \tag{6.7}$$

The corresponding expression for a ring vortex is obtained by making the substitution $\pi r_0^2 j = I \rightarrow r_0^2 c\omega/2$, where r_0 is the radius of the vortex core.

The moment of a ring vortex therefore is

$$m = (1/2)r_0^2\omega\pi R^2 \tag{6.8}$$

Inserted into (6.4), one obtains a corresponding nonlinear term quadratic in the vector potentials of the vortices.

The hierarchical displacement and analogies to hydrodynamics are made complete by recognizing that the gauge function f is related to the velocity potential of an irrotational flow. A gauge transformation leaving the forces unchanged corresponds in the hydrodynamic picture to the addition of an irrotational flow field. These analogies and hierarchical displacements of Newtonian point mechanics and Einstein gravity versus Helmholtz's vortex dynamics and Yang–Mills gauge field theories are shown in Table I.

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